INTEGRATION OF ONE-FORMS ON p-ADIC ANALYTIC SPACES

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Recall that there is a unique way to define for every complex manifold X, every closed analytic one-form ω, and every continuous path $\gamma : [0,1] \to X$, a number $\int_\gamma \omega \in \mathbf{C}$, called the integral of ω along γ, such that the following is true:

(a) if $\omega = df$ for an analytic function f on X, then $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$;

(b) $\int_\gamma \omega$ depends only on the homotopy class of γ;

(c) given a second path $\sigma : [0,1] \to X$ with $\sigma(0) = \gamma(1)$, one has $\int_{\sigma\circ\gamma} \omega = \int_\gamma \omega + \int_\sigma \omega$.

The definition is based on the facts that each point of X has an open neighborhood isomorphic to an open polydisc and each closed form on the latter is exact (Poincaré lemma). Namely, if $\gamma([0,1]) \subset \mathcal{U}_1 \cup ... \cup \mathcal{U}_n$ and $t_1 = 0 < t_2 < ... < t_{n+1} = 1$ are such that each $\mathcal{U}_i$ is isomorphic to an open polydisc and $\gamma([t_i, t_{i+1}]) \subset \mathcal{U}_i$ for all $1 \le i \le n$, then $\int_\gamma \omega = \sum_{i=1}^n (f_i(\gamma(t_{i+1})) - f_i(\gamma(t_i)))$, where f_i is a primitive of ω at $\mathcal{U}_i$. (Of course, one checks that the integral depends only on the homotopy class of the path.)

Can an integral of a closed one-form along a path be defined for separated smooth analytic spaces X over a non-Archimedean field k of characteristic zero?

That such definition is possible was demonstrated by Robert Coleman for smooth k-analytic curves ([Col], CoSh]) with k a closed subfield of $\mathbf{C}_p$, the completion of an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. I am going to explain the results from [Ber] which extend Coleman's work to smooth k-analytic spaces of arbitrary dimension.

First of all, we recall that each point $x \in X$ has an associated non-Archimedean field $\mathcal{H}(x)$ so that the value $f(x)$ of a function f analytic in an open neighborhood of x lies in $\mathcal{H}(x)$. The field $\mathcal{H}(x)$ is in general bigger than k, and so an integral along a path $\gamma : [0,1] \to X$ can be defined at least under the assumption that the ends $\gamma(0)$ and $\gamma(1)$ of γ are k-rational points, i.e., $\mathcal{H}(x) = k$. Such a point has a sufficiently small open neighborhood isomorphic to an open polydisc, and the classical Poincaré lemma implies that every closed one-form ω in an open neighborhood of the point has a primitive (in a smaller neighborhood). Moreover, the latter fact holds for a point $x \in X$ if and only if x has a fundamental system of étale neighbourhoods isomorphic to an open polydisc or, equivalently, the residue field $\widetilde{\mathcal{H}(x)}$ is algebraic over $\widetilde{k}$ and the group $|\mathcal{H}(x)^*|/|k^*|$ is torsion.

Let X_{st} denote the set of points with the above property. Although the set X_{st} is dense in X, it is much smaller than X so that the topology on X_{st} induced from that on X is totally disconnected. This means that any path connecting two distinct k-rational points contains points at which closed analytic one-forms non necessarily have a primitive in the class of analytic functions. Thus, in order to define an integral of a closed one-form along a path, one should define a primitive of that form in an open neighborhood of each point of X in a bigger class of functions.

Furthermore, if this is done, one may ask how to define a primitive of a closed one-form with coefficients in the module over the ring of analytic functions generated by the previous primitives, and so on. What is clear is that all primitive constructed in such a way should be analytic in an open neighsorhood of each point from X_{st}.

To make things more precise, let us consider the simplest example of a closed one-form that has no a primitive in the class of analytic functions. Let X be the analytification $\mathbf{G}_{\mathrm{m}} = \mathbf{A}^1 \backslash \{0\}$ of the one dimensional split torus. Then the one-form $\frac{dT}{T}$ has no primitive at any open neighborhood of each point of the skeleton $S(\mathbf{G}_{\mathrm{m}})$, i.e., the maximal point of a closed disc with centre at zero. This one-form has an analytic primitive at the open disc with centre at any k-rational point $a \in \mathbf{G}_{\mathrm{m}}(k) = k^*$ of radius $|a|$. For example, if $a = 1$, the power series of the usual logarithm $-\sum_{i=1}^{\infty} \frac{(1-T)^i}{i}$ is such a primitive. The natural requirement for a global primitive f that coincides with the latter at the open disc of radius one with centre at one is that it should behave functorially with respect to $\frac{dT}{T}$. If m and p_i, $i = 1, 2$, denote the multiplication and i-th projection $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$, then $m^*(\frac{dT}{T}) = p_1^*(\frac{dT}{T}) = p_2^*(\frac{dT}{T})$. Thus, the primitive f should satisfy the relation $m^*(f) = p_1^*(f) + p_2^*(f)$ which in the usual form is written as $f(ab) = f(a) + f(b)$. Such a primitive is determined by its value at p. Fixing this value λ, we get a primitive $\mathrm{Log}^{\lambda}(T)$ (called a branch of the logarithm) that satisfies the above relation and is analytic at the complement of the skeleton $S(\mathbf{G}_{\mathrm{m}})$.

In [Ber], we consider a general situation that allows one to consider all possible branches of the logarithm. Namely, we fix a filtered k-algebra K, i.e., a commutative k-algebra provided with an exhausting filtration by k-vector subspaces $K^0 \subset K^1 \subset \ldots$ with $K^i \cdot K^j \subset K^{i+j}$, and an element $\lambda \in K^1$ which is assigned to be the value of the logarithm at p. For example, a branch of the logarithm considered in Coleman's work is obtained for $K = k$. If K is the ring of polynomials $k[\log(p)]$ in the variable $\log(p)$ with $K^i = k[\log(p)]_{\leq i}$, the subspace of polynomials of degree at most i, and $\lambda = \log(p)$, the corresponding primitive is denoted by $\mathrm{Log}(T)$ and called the universal logarithm.

We now introduce the maximal class of functions that contains all possible primitives. For $i \geq 0$, we set $\mathcal{O}_X^{K,i} = \mathcal{O}_X \otimes_k K^i$ and denote by $\mathfrak{N}^{K,i}$ the étale sheaf on X defined as follows: for an étale morphism $Y \to X$, $\mathfrak{N}^{K,i}(Y) = \varinjlim \mathcal{O}^{K,i}(\mathcal{V})$, where the inductive limit is taken over all open neighborhoods $\mathcal{V}$ of Y_{st} in Y. The inductive limit $\mathfrak{N}_X^K = \varinjlim \mathfrak{N}_X^{K,i}$ is a sheaf of filtered $\mathcal{O}_X$-algebras, called the sheaf of naive analytic functions. The sheaves $\mathfrak{N}_X^{K,i}$ are functorial with respect to X and, in particular, for a morphism $\varphi : Y \to X$ and a function $f \in \mathfrak{N}^{K,i}(X)$, there is a well defined function $\varphi^\sharp(f) \in \mathfrak{N}^{K,i}(Y)$.

Furthermore, for $q \geq 0$, the sheaf of $\mathfrak{N}^{K,i}$-differential q-forms $\Omega^i_{\mathfrak{N}^{K,i},X}$ is the tensor product $\mathfrak{N}_X^{K,i} \otimes_{\mathcal{O}_X} \Omega_X^q$. The differentials $d : \Omega_X^q \to \Omega_X^{q+1}$ induce differentials $d : \Omega^q_{\mathfrak{N}^{K,i},X} \to \Omega^{q+1}_{\mathfrak{N}^{K,i},X}$. Notice that the kernel of the first differential $d : \mathfrak{N}_X^{K,i} \to \Omega^1_{\mathfrak{N}^{K,i},X}$ is much bigger than the sheaf $\mathcal{C}_X^{K,i} = \mathfrak{c}_X \otimes_k K^i$, where $\mathfrak{c}_X = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{d} \Omega_X^1)$. (The latter is called the sheaf of constant analytic functions; if k is algebraically closed, it is the constant sheaf associated to k.)

A $\mathcal{D}_X$-module is an étale $\mathcal{O}_X$-module $\mathcal{F}$ provided with an integrable connection $\nabla : \mathcal{F} \to \Omega^1_{\mathcal{F}} = \mathcal{F} \otimes_{\mathcal{O}_X} \Omega^1_X$. A $\mathcal{D}_X$-algebra is an étale commutative $\mathcal{O}_X$-algebra

$\mathcal{A}$ which is also a $\mathcal{D}_X$-module whose connection satisfies the Leibniz rule $\nabla(fg) = fdg + gdf$. If in addition $\mathcal{A}$ is a filtered $\mathcal{O}_X$-algebra such that all $\mathcal{A}^i$ are $\mathcal{D}_X$-submodules of $\mathcal{A}$, then $\mathcal{A}$ is said to be a filtered $\mathcal{D}_X$-algebra. For example, $\mathfrak{N}_X$ is a filtered $\mathcal{D}_X$-algebra. Here is the main result of [Ber].

Theorem 1. *Given a closed subfield $k \subset \mathbf{C}_p$, a filtered k-algebra K and an element $\lambda \in K^1$, there is a unique way to provide every separated smooth k-analytic space X with a filtered $\mathcal{D}_X$-subalgebra $\mathcal{S}^\lambda_X \subset \mathfrak{N}^K_X$ such that the following is true:*

(a) $\mathcal{S}^{\lambda,0}_X = \mathcal{O}_X \otimes_k K^0$;

(b) $\mathrm{Ker}(\mathcal{S}^{\lambda,i}_X \stackrel{d}{\to} \Omega^1_{\mathcal{S}^{\lambda,i},X}) = \mathcal{C}^{K,i}_X$;

(c) $\mathrm{Ker}(\Omega^1_{\mathcal{S}^{\lambda,i},X} \stackrel{d}{\to} \Omega^2_{\mathcal{S}^{\lambda,i},X}) \subset d\mathcal{S}^{\lambda,i+1}_X$;

(d) $\mathcal{S}^{\lambda,i+1}_X$ *is generated by local sections f such that* $df \in \Omega^1_{\mathcal{S}^{\lambda,i},X}$;

(e) $\mathrm{Log}^\lambda(T) \in \mathcal{S}^{\lambda,1}(\mathbf{G}_\mathrm{m})$;

(f) for any morphism $\varphi : X' \to X$, one has $\varphi^\sharp(\mathcal{S}^{\lambda,i}_X) \subset \mathcal{S}^{\lambda,i}_{X'}$.

The sheaves $\mathcal{S}^\lambda_X$ possess many additional properties. We mention only that, if X is connected, then for any nonempty open subset $\mathcal{U} \subset X$ the homomorphism $\mathcal{S}^\lambda(X) \to \mathcal{S}^\lambda(\mathcal{U})$ is injective. The sheaves $\mathcal{S}^\lambda_X$ are functorial with respect to k, X, K and λ. For example, given a homomorphism of filtered k-algebras $K \to K' : \lambda \mapsto \lambda'$, one has $\mathcal{S}^\lambda_X \otimes_K K' \widetilde{\to} \mathcal{S}^{\lambda'}_X$. Furthermore, the sheaves $\mathcal{S}^\lambda_X$ are much bigger than the sheaf $\mathcal{O}_X \otimes_k K$. For example, let $\mathfrak{s}^i_X$ denote the subsheaf of $\mathcal{S}^{\log(p),i}_X$ (for the universal logarithm) consisting of the functions f that do not depend on $\log(p)$, i.e., the restriction of f to some open neighborhood $\mathcal{U}$ of each point $x \in X_{st}$ belongs to $\mathcal{O}(\mathcal{U})$. Then for every $i \geq 1$, the quotient $\mathfrak{s}^i(\mathbf{P}^1)/\mathfrak{s}^{i-1}(\mathbf{P}^1)$ is of infinite dimension over k.

Theorem 1 is used to construct the required integrals of closed one-forms along a path. For a k-analytic space X, we set $\overline{X} = X\widehat{\otimes}_k\mathbf{C}_p$, and denote by $H_1(X, \mathbf{Q})$ and $H_1(\overline{X}, \mathbf{Q})$ the singular homology of X and $\overline{X}$ with rational coefficients.

Theorem 2. *Given (k, K, λ) as in Theorem 1, there is a unique way to construct, for every separated smooth k-analytic space X with $H_1(\overline{X}, \mathbf{Q}) \widetilde{\to} H_1(X, \mathbf{Q})$, every closed one-form $\omega \in \Omega^1_{\mathcal{S}^\lambda,i}(X)$ and every path $\gamma : [0, 1] \to X$ with ends in $X(k)$, an integral $\int_\gamma \omega \in K^{i+1}$ such that the following is true:*

(a) if $\omega = df$ for $f \in \mathcal{S}^{\lambda,i+1}(X)$, then $\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$;

(b) $\int_\gamma \omega$ depends only on the homotopy class of γ;

(c) given a second path $\sigma : [0, 1] \to X$ with ends in $X(k)$ and $\sigma(0) = \gamma(1)$, one has $\int_{\sigma\circ\gamma} \omega = \int_\gamma \omega + \int_\sigma \omega$.

One shows that the integral $\int_\gamma \omega$ is linear on ω, functorial on X, and depends nontrivially on the homotopy class of γ. Moreover, if $\gamma([0, 1]) \subset Y$, where Y is an analytic subdomain of X with good reduction, then $\int_\gamma \omega \in K^0$.

The next natural question is as follows. What are the analytic differential equations which have a full set of solutions in the bigger class of functions $\mathcal{S}_X$?

Let $\mathcal{F}$ be an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. (Recall that such $\mathcal{F}$ is always a locally free $\mathcal{O}_X$-module.) $\mathcal{F}$ is said to be trivial, if it is isomorphic to a direct sum of copies of the $\mathcal{D}_X$-module $\mathcal{O}_X$. It is said to be unipotent if there is a sequence of $\mathcal{D}_X$-submodules $\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset ... \subset \mathcal{F}^n = \mathcal{F}$ such that all quotients $\mathcal{F}^i/\mathcal{F}^{i-1}$ are trivial $\mathcal{D}_X$-modules. The minimal n with this property is called the level of $\mathcal{F}$.

Finally, $\mathcal{F}$ is said to be locally (resp. étale locally) unipotent, if every point has an open (resp. étale) neighborhood U such that $\mathcal{F}\big|_U$ is unipotent.

Theorem 3. *Given $x \in X$ and $n \geq 1$, the following are equivalent:*

(a) there is an étale neighborhood $U \to X$ of x such that $\mathcal{F}\big|_U$ is unipotent of level at most n;

(b) there is an étale neighborhood $U \to X$ of x such that, for some $m \geq 1$, there is an embedding of $\mathcal{D}_U$-modules $\mathcal{F}\big|_U \hookrightarrow (\mathcal{S}_U^{\lambda,n-1})^m$.

Let $\mathcal{F}_{\mathcal{S}^\lambda}$ be the $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{S}_X^\lambda$. It is a $\mathcal{D}_{\mathcal{S}^\lambda}$-module, i.e., a sheaf of $\mathcal{D}$-modules over the $\mathcal{D}_X$-algebra $\mathcal{S}_X^\lambda$. The sheaf of horizontal sections $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla$ is an étale sheaf of $\mathcal{C}_X^K$-modules, where $\mathcal{C}_X^K = \mathfrak{c}_X \otimes_k K$.

Corollary 4. *The following properties of $\mathcal{F}$ are equivalent:*

(a) $\mathcal{F}$ is étale locally unipotent;

(b) the étale $\mathcal{C}_X^K$-module $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla$ is locally free;

(c) there is an isomorphism of $\mathcal{D}_{\mathcal{S}^\lambda}$-modules $\mathcal{F}_{\mathcal{S}^\lambda}^\nabla \otimes_{\mathcal{C}_X^K} \mathcal{S}_X^\lambda \widetilde{\to} \mathcal{F}_{\mathcal{S}^\lambda}$.

Notice that the stalk of $\mathcal{F}_{\mathcal{S}}^\nabla$ at a point $x \in X_{st}$ coincides with $\mathcal{F}_x^\nabla \otimes_k K$. This allows one to define a parallel transport $T_\gamma^{\mathcal{F}} : \mathcal{F}_x^\nabla \otimes_k K \widetilde{\to} \mathcal{F}_y^\nabla \otimes_k K$ along a path $\gamma : [0,1] \to X$ with ends in $x, y \in X_{st}$ (see [Ber, §9.4]).

Let now X be a smooth k-analytic curve of the form $\mathcal{X}^{\mathrm{an}} \setminus \coprod_{i=1}^n E_i$, where $\mathcal{X}$ is a smooth projective curve over k with good reduction, and E_i are affinoid subdomains isomorphic to a closed disc in $\mathbf{A}^1$ with center at zero and lying in pairwise distinct residue classes of the reduction. (It is what Coleman calls a basic wide open curve.) Such X is simply connected and $H^1(X, \mathfrak{c}_X) = 0$. It follows that each one-form $\omega \in \Omega^1_{\mathcal{S}^\lambda,i}(X)$ has a primitive in $\mathcal{S}^{\lambda,i+1}(X)$. We set $A^{\lambda,0}(X) = \mathcal{O}(X) \otimes_k K^0$ and, for $i \geq 1$, define $A^{\lambda,i}(X)$ as the $\mathcal{O}(X)$-submodule of $\mathcal{S}^{\lambda,i}(X)$ generated by primitives of one-forms from $A^{\lambda,i-1}(X) \otimes_{\mathcal{O}(X)} \Omega^1(X)$. The filtered $\mathcal{O}(X)$-algebra of locally analytic functions constructed by Coleman is $A^\lambda(X) = \bigcup_{i=0}^\infty A^{\lambda,i}(X)$ for $K = k$. Since X is simply connected, the integral of a one-form with coefficients in $A(X)$ along a path $\gamma : [0,1] \to X$ with ends in $X(k)$ depends only on the ends, and it coincides with the integral constructed by Coleman.

The proof of Theorem 1 follows ideas of Coleman's construction. The Frobenius endomorphism, which is used even in the formulation of his result, is sewn here in the proof. Among main ingredients are a local description of smooth k-analytic spaces, based on de Jong's alteration results, and the fact that each point of a smooth k-analytic space has a fundamental system of open neighborhoods $\mathcal{U}$ such that $H^1(\mathcal{U}, \mathfrak{c}_{\mathcal{U}}) = 0$.

Theorem 1 implies that the de Rham complex $0 \to \mathcal{C}_X^K \to \mathcal{S}_X^\lambda \to \Omega^1_{\mathcal{S}^\lambda,X} \to \dots$ is exact at $\Omega^q_{\mathcal{S}^\lambda,X}$ for $q = 0, 1$, and we conjecture that it is exact for all q.

One may ask if something like Theorems 1 and 2 holds for other non-Archimedean fields of characteristic zero. Of course, for this one should impose additional properties on the integral. Here is a simple example for the field of complex numbers $\mathbf{C}$ provided with the trivial valuation.

Let $\mathcal{X}$ be an irreducible separated smooth scheme of finite type over $\mathbf{C}$. Then the (non-Archimedean) analytification $\mathcal{X}^{\mathrm{an}}$ of $\mathcal{X}$ is a contractible topological space. This implies that a possible integral $\int_\gamma \omega$ of a closed one-form $\omega \in \Omega^1(\mathcal{X})$ along a path $\gamma : [0,1] \to \mathcal{X}^{\mathrm{an}}$ with ends $x, y \in \mathcal{X}(\mathbf{C})$ should depend only on the ends x, y, and so it can be denoted by $\int_x^y \omega$. Let us require that the map $\mathcal{X}(\mathbf{C}) \to$

$\mathbf{C} : y \mapsto \int_x^y \omega$ is continuous in the complex topology of both spaces. This imposes continuity on the logarithm homomorphism $\mathbf{C}^* \to \mathbf{C}$. Any such homomorphism corresponds to a complex number $\lambda \in \mathbf{C}$ with $|\lambda| \neq 1$. Namely, if $\lambda = re^{i\varphi} \neq 0$, the corresponding homomorphism $\log^\lambda : \mathbf{C}^* \to \mathbf{C}$ takes $z \in \mathbf{C}^*$ to $\log_r(|z|)e^{i\varphi}$, and if $\lambda = 0$, it takes every z to zero. An analog of Theorem 2 for algebraic one-forms states that, given $\lambda \in \mathbf{C}$ with $|\lambda| \neq 1$, there is a unique way to construct for every irreducible separated smooth scheme of finite type over $\mathbf{C}$, any closed one-forms $\omega \in \Omega^1(\mathcal{X})$ and any pair of points $x, y \in \mathcal{X}(\mathbf{C})$, an integral $\int_x^y \omega \in \mathbf{C}$ such that the following is true:

(a) if $\omega = df$ for $f \in \mathcal{O}(\mathcal{X})$, then $\int_x^y \omega = f(y) - f(x)$;

(b) for a third point $z \in \mathcal{X}(\mathbf{C})$, one has $\int_x^z \omega = \int_x^y \omega + \int_y^z \omega$;

(c) $\int_\gamma \omega$ is linear on ω;

(d) for any point $z \in \mathbf{C}^*$, one has $\int_1^z \frac{dT}{T} = \log^\lambda(z)$;

(e) the map $\mathcal{X}(\mathbf{C}) \to \mathbf{C} : y \mapsto \int_x^y \omega$ is continuous in the complex topology;

(f) for any morphism $\varphi : \mathcal{X}' \to \mathcal{X}$ and any pair of points $x', y' \in \mathcal{X}'(\mathbf{C})$, one has $\int_{x'}^{y'} \varphi^*(\omega) = \int_{\varphi(x')}^{\varphi(y')} \omega$.

If $\mathcal{X}$ is proper, the integral $\int_x^y \omega$ is always zero, which is not surprising since it depends on the ends of a path only. Furthermore, let $\widehat{\mathcal{X}}$ be $\mathcal{X}$ considered as a formal scheme. Its generic fiber $\widehat{\mathcal{X}}_\eta$ is a closed analytic domain in $\mathcal{X}^{\mathrm{an}}$ (which coincides with $\mathcal{X}^{\mathrm{an}}$ if $\mathcal{X}$ is proper), and let π be the reduction map $\widehat{\mathcal{X}}_\eta \to \widehat{\mathcal{X}}_s = \mathcal{X}$. Since $\mathcal{X}$ is smooth, the preimage $\pi^{-1}(x)$ of each point $x \in \mathcal{X}(\mathbf{C})$ is isomorphic to the (non-Archimedean) unit open polydisc with centre at zero with the only $\mathbf{C}$-rational point x. Each closed one-form $\omega \in \Omega^1(\mathcal{X})$ has a primitive f_x at $\pi^{-1}(x)$ defined up to a constant. If we fix a point $x_0 \in \mathcal{X}(\mathbf{C})$, the analytic function f on $\pi^{-1}(\mathcal{X}(\mathbf{C}))$, whose restriction to $\pi^{-1}(x)$ is the local primitive f_x with $f_x(x) = \int_{x_0}^x \omega$, is a global primitive of ω which does not depend of x_0 up to a constant. It would be interesting to know if it is possible to iterate this procedure, i.e., to construct a primitive of a closed one-form with coefficients in the $\mathcal{O}(\mathcal{X})$-module generated by the above primitives f's, and so on.

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